

Scattering solutions and Born approximation for the magnetic Schrödinger operator

Valery Serov¹ and Jan Sandhu²

Department of Mathematical Sciences
University of Oulu, Finland

Abstract

We prove the existence of scattering solutions for multidimensional magnetic Schrödinger equation which belong to the weighted Sobolev space $H_{-\delta}^1(R^n)$ ($n = 2, 3$) with some $\delta > \frac{1}{2}$. As a consequence of this we formulate the direct Born approximation for the magnetic Schrödinger operator. Possible connections with inverse problems (inverse scattering Born approximation) are discussed.

1 Introduction

The main goal of present article is to justify the application of the classical direct scattering Born approximation for the magnetic Schrödinger operator. The direct Born approximation is known as the most applicable approximate method in the numerous practical problems. It is also known that the inverse scattering Born approximation is well-defined and perfectly works (as the mathematical tool) in the case of linear and nonlinear Schrödinger operators and for all types of scattering data: full scattering, backscattering, fixed angle scattering and fixed energy scattering. For some scattering data it is possible to get the uniqueness and reconstruction procedure while for some data we are able to reconstruct singularities and jumps of unknowns even when there is no uniqueness. We mention here the results of Päivärinta and Somersalo [10], Nachman [7], [8], Sun and Uhlmann [26], Isakov and Sylvester [4], Päivärinta, Serov and Somersalo [15], Päivärinta and Serov [11], [13], [14], Ola, Päivärinta and Serov [9], Ruiz [18], Ruiz and Vargas [19], Päivärinta and Serov [12], Reyes [16], Serov [20], Serov and Harju [21], [22], Serov and

¹E-mail address: vserov@cc.oulu.fi

²E-mail address: jan.sandhu@oulu.fi

Sandhu [23], Lechleiter [6], Reyes and Ruiz [17] and some others. The main point of all these results is the precise calculation of the first (quadratic) nonlinear term in the Born series. For the magnetic Schrödinger operator the direct scattering problem (i.e., existence of the scattering solutions) as well as the inverse scattering Born approximation are not familiar at all. The big interest to this problem is connected to the fact that the knowledge of the scattering amplitude with backscattering data allows us to obtain essential information about the unknowns.

We consider the magnetic Schrödinger operator

$$H = -(\nabla + i\vec{W}(x))^2 + V(x), \quad x \in R^n, \quad (1.1)$$

in dimensions $n = 2, 3$, where the coefficients $\vec{W}(x)$ and $V(x)$ are assumed to be real-valued. We assume generally that $\vec{W}(x) \in L^\infty_\delta(R^n)$ and

$$\nabla \vec{W}(x) \in L^p_\delta(R^n), \quad n = 3, \quad 3 \leq p \leq \infty; \quad n = 2, \quad 2 < p \leq \infty \quad (1.2)$$

and

$$V(x) \in L^p_\delta(R^n), \quad n = 3, \quad 3 \leq p \leq \infty; \quad n = 2, \quad 2 < p \leq \infty, \quad (1.3)$$

where $\delta > \frac{n+1}{2} - \frac{n}{p}$. Here L^p_σ denotes usual weighted Lebesgue space and Sobolev space $W^1_{p,\sigma}$ is understood so that f belongs to $W^1_{p,\sigma}(R^n)$ if and only if f and ∇f belong to $L^p_\sigma(R^n)$. For the case $p = 2$ instead of the symbol $W^1_{2,\sigma}$ we use the symbol H^1_σ .

It is well-known that under these conditions for the coefficients of the magnetic Schrödinger operator the following Gårding's inequality holds:

$$(Hu, u)_{L^2(R^n)} \geq \nu \|\nabla u\|_{L^2(R^n)}^2 - C \|u\|_{L^2(R^n)}^2,$$

where $0 < \nu < 1, C > 0$. This inequality allows us to define symmetric operator H by the method of quadratic forms. H has a self-adjoint Friedrichs extension with the domain (in general)

$$D(H) = \{f(x) \in W^1_2(R^n) : Hf(x) \in L^2(R^n)\}.$$

In our particular case it is possible to prove that actually

$$D(H) = W^2_2(R^n).$$

In the scattering theory the main role are played by the special solutions of the equation

$$Hu(x) = k^2 u(x)$$

which are of the form

$$u(x) = u_0(x) + u_{sc}(x),$$

where $u_0(x) = e^{ik(x,\theta)}$ is incident wave with direction $\theta \in S^{n-1}$ and the scattered wave $u_{sc}(x)$ satisfies the Sommerfeld radiation condition at the infinity, i.e.

$$\lim_{r \rightarrow +\infty} r^{\frac{n-1}{2}} \left(\frac{\partial u_{sc}(x)}{\partial r} - ik u_{sc}(x) \right) = 0, \quad r = |x|. \quad (1.4)$$

In this case the total field u satisfies the so-called Lippmann-Schwinger equation

$$u = u_0 + \int_{R^n} G_k^+(|x-y|) \left(i \nabla(\vec{W}(y)u) + i \vec{W}(y) \nabla u - \tilde{q}(y)u \right) dy, \quad (1.5)$$

where $\tilde{q} = |\vec{W}|^2 + V$ and G_k^+ is the kernel of integral operator $(-\Delta - k^2 - i0)^{-1}$. Using the representation $u = u_0 + u_{sc}$ we rewrite this integral equation (1.5) only for scattered field u_{sc} as

$$u_{sc} = \tilde{u}_0 + \int_{R^n} G_k^+(|x-y|) \left(i \nabla(\vec{W}(y)u_{sc}) + i \vec{W}(y) \nabla u_{sc} - \tilde{q}(y)u_{sc} \right) dy, \quad (1.6)$$

where \tilde{u}_0 is equal to

$$\tilde{u}_0(x) = \int_{R^n} G_k^+(|x-y|) \left(i \nabla(\vec{W}(y)u_0) + i \vec{W}(y) \nabla u_0 - \tilde{q}(y)u_0 \right) dy. \quad (1.7)$$

We use the following results of Agmon [2] (see Remark 2, Appendix A):

$$\frac{1}{|k|} \|f\|_{H_{-\delta}^2(R^n)} + \|f\|_{H_{-\delta}^1(R^n)} + |k| \|f\|_{L_{-\delta}^2(R^n)} \leq c \|(\Delta + k^2)f\|_{L_{\delta}^2(R^n)}, \quad |k| \geq 1,$$

where $\delta > \frac{1}{2}$ and $H_{-\delta}^2(R^n)$ denotes the weighted Sobolev space. As a consequence we have (for fixed k) that

$$\|(-\Delta - k^2 - i0)^{-1}f\|_{H_{-\delta}^2(R^n)} \leq c(k) \|f\|_{L_{\delta}^2(R^n)},$$

and uniformly in $|k| \geq 1$ we have that

$$\|(-\Delta - k^2 - i0)^{-1}f\|_{L_{-\delta}^2(R^n)} \leq \frac{c}{|k|} \|f\|_{L_{\delta}^2(R^n)}. \quad (1.8)$$

But since $(-\Delta - k^2 - i0)^{-1}$ is the integral operator of convolution type we can conclude that for fixed k it maps continuously $H_{\delta}^{-1}(R^n)$ to $H_{-\delta}^1(R^n)$, where $H_{\delta}^{-1}(R^n)$ denotes the dual of the Sobolev space $H_{-\delta}^1(R^n)$.

We rewrite (1.6) as the integral equation

$$u_{sc} = \tilde{u}_0 + L_k(u_{sc}), \quad \tilde{u}_0 = L_k(u_0),$$

where the integral operator L_k is defined as

$$L_k f(x) := \int_{\mathbb{R}^n} G_k^+(|x-y|) \left(i \nabla(\vec{W}(y)f) + i \vec{W}(y) \nabla f - \tilde{q}(y)f \right) dy. \quad (1.9)$$

The main result of present article is Theorem 2.1 which provides the existence of the scattering solutions for the magnetic Schrödinger operator. This theorem is proved in Section 2. Based on the main result we justify in Section 3 the direct Born approximation for such operators.

2 Existence of the scattering solutions

We are preceding a proof of the main result by the following lemmas.

Lemma 2.1. *Suppose that conditions (1.2) and (1.3) are fulfilled. Then there is $\delta_0 > \frac{1}{2}$ such that $\tilde{u}_0 \in H_{-\delta_0}^1(\mathbb{R}^n)$ and the integral operator L_k maps $H_{-\delta_0}^1(\mathbb{R}^n)$ into itself.*

Proof. Conditions for p and δ from (1.2) and (1.3) imply that there is $\delta_0 > \frac{1}{2}$ such that

$$L_\delta^p(\mathbb{R}^n) \subset L_{\delta_0}^2(\mathbb{R}^n).$$

It is therefore true that under the conditions (1.2) and (1.3) functions V , \vec{W} , $\nabla \vec{W}$ and $|\vec{W}|^2$ belong to $L_{\delta_0}^2(\mathbb{R}^n)$ with the same δ_0 . Since u_0 is a bounded and smooth function we may conclude (using Agmon's result (1.8)) that \tilde{u}_0 belongs to $H_{-\delta_0}^1(\mathbb{R}^n)$. It can be mentioned here that we have no longer uniform estimates in k as in (1.8). In order to prove that L_k maps $H_{-\delta_0}^1(\mathbb{R}^n)$ into itself we note that if f belongs to $H_{-\delta_0}^1(\mathbb{R}^n)$ then $(1+|x|^2)^{-\frac{\delta_0}{2}} f$ belongs to usual Sobolev space $H^1(\mathbb{R}^n)$. Using now Sobolev imbedding theorem we conclude that

$$f \in L_{-\delta_0}^{\frac{2n}{n-2}}(\mathbb{R}^n), \quad n = 3, \quad f \in L_{-\delta_0}^s(\mathbb{R}^n), \quad s < \infty, \quad n = 2.$$

Then the conditions (1.2) and (1.3) and Hölder inequality allow us easily conclude that $\tilde{q}f$ and $(\nabla \vec{W})f$ belong $L_{\delta_0}^2(\mathbb{R}^n)$. Since we have $\vec{W}(x) \in L_\delta^\infty(\mathbb{R}^n)$ the function $\vec{W} \nabla f$ will belong to $L_{\delta_0}^2(\mathbb{R}^n)$ too. The final step is the application of Agmon's result (1.8).

We may prove a little bit more about this operator L_k .

Lemma 2.2. *Let us assume that $\vec{W} \in L^\infty_\delta(R^n)$,*

$$\nabla \vec{W}(x) \in L^p_\delta(R^n), \quad n = 3, \quad 3 \leq p \leq \infty; \quad n = 2, \quad 2 < p \leq \infty, \quad (2.1)$$

where $\delta > \frac{n+1}{2} - \frac{n}{p}$, and

$$V(x) \in L^p_{loc}(R^n), \quad n = 3, \quad 3 \leq p \leq \infty; \quad n = 2, \quad 2 < p \leq \infty, \quad (2.2)$$

and that \vec{W} and V have special behavior at the infinity such that

$$|V(x)|, \quad |\vec{W}(x)|, \quad |\nabla \vec{W}(x)| \leq \frac{c}{|x|^\mu}, \quad |x| \rightarrow \infty, \quad (2.3)$$

where $\mu > 2$ for $n = 2, 3$. Then the operator L_k is compact in $H^1_{-\delta_0}(R^n)$ for some $\delta_0 > \frac{1}{2}$.

Proof. Let us choose $R > 0$ large enough and represent V, \vec{W} and $\nabla \vec{W}$ as

$$V = V_1 + V_2, \quad \vec{W} = \vec{W}_1 + \vec{W}_2, \quad \nabla \vec{W} = \nabla \vec{W}_1 + \nabla \vec{W}_2,$$

where the supports of the functions V_1, \vec{W}_1 and $\nabla \vec{W}_1$ are included in the ball $B_R = \{x \in R^n : |x| \leq R\}$, but supports of the functions V_2, \vec{W}_2 and $\nabla \vec{W}_2$ are included in the set $\{x \in R^n : |x| \geq R\}$. Without loss of generality we assume in addition that the functions V_2, \vec{W}_2 and $\nabla \vec{W}_2$ are continuous and satisfy the conditions (2.3) for all $|x| \geq R$.

Conditions (2.1)-(2.2) and the previous considerations imply that

$$i\nabla(\vec{W}_1(x)f(x)) + i\vec{W}_1(x)\nabla f(x) - \tilde{q}_1(x)f(x) \in L^2(B_R)$$

for any function $f \in H^1_{-\delta_0}(R^n)$. But $L^2(B_R)$ is compactly imbedded in $H^{-1}(B_R)$ and therefore in $H^{-1}_{\delta_0}(R^n)$ for $\delta_0 > \frac{1}{2}$. It remains to mention now that due to Agmon's result (1.8) operator $(-\Delta - k^2 - i0)^{-1}$ maps continuously the space $H^{-1}_{\delta_0}(R^n)$ to the space $H^1_{-\delta_0}(R^n)$. Thus, part of the operator L_k that corresponds to V_1, \vec{W}_1 and $\nabla \vec{W}_1$ is compact in $H^1_{-\delta_0}(R^n)$ for $\delta_0 > \frac{1}{2}$. Since outside the ball B_R these functions V_2, \vec{W}_2 and $\nabla \vec{W}_2$ satisfy the conditions (2.3) we may firstly conclude using Sobolev imbedding theorem that for $f \in H^1_{-\delta_0}(R^n)$ we have

$$i\nabla(\vec{W}_2(x)f(x)) + i\vec{W}_2(x)\nabla f(x) - \tilde{q}_2(x)f(x) \in L^2_{\delta_0}(R^n)$$

for some δ_0 if and only if $\mu > 2\delta_0 + 1$. But under the conditions of Lemma 2.2 this criterion is satisfied if $\delta_0 > \frac{1}{2}$ is chosen appropriately.

Since the conditions (2.3) are satisfied we can find two sequences $\phi_j(x) \in C_0^\infty(R^n \setminus B_R)$ and $\psi_j(x)$ (vector-valued) $\in C_0^\infty(R^n \setminus B_R)$ such that

$$\|\phi_j - V_2\|_{L_{\delta'}^\infty(R^n \setminus B_R)} \rightarrow 0, \quad \|\psi_j - \vec{W}_2\|_{L_{\delta'}^\infty(R^n \setminus B_R)} \rightarrow 0,$$

$$\|\nabla \psi_j - \nabla \vec{W}_2\|_{L_{\delta'}^\infty(R^n \setminus B_R)} \rightarrow 0$$

as $j \rightarrow \infty$ for any $\delta' < \mu$. These approximation properties imply that

$$\begin{aligned} & \|i\nabla((\vec{W}_2 - \psi_j)f) + i(\vec{W}_2 - \psi_j)\nabla f - (\tilde{q}_2 - Q_j)f\|_{L_{\delta_0}^2(R^n)} \leq \\ & \leq c\|\nabla(\vec{W}_2 - \psi_j)\| + \|\vec{W}_2 - \psi_j\| + \|\tilde{q}_2 - Q_j\|_{L_{2\delta_0}^\infty(R^n)}\|f\|_{H_{-\delta_0}^1(R^n)} \rightarrow 0 \end{aligned} \quad (2.4)$$

as $j \rightarrow \infty$, where $Q_j = |\psi_j|^2 + \phi_j$. Since we can choose $\delta_0 > \frac{1}{2}$ and μ from condition (2.3) such that $2\delta_0 < \mu$ then (2.4) means that part of the operator L_k which corresponds to V_2, \vec{W}_2 and $\nabla \vec{W}_2$ is compact in $H_{-\delta_0}^1(R^n)$ too.

Lemma 2.3. *Under the same assumptions as in Lemma 2.2 for any fixed $k > 0$ and for any $f \in H_{-\delta_0}^1(R^n)$ with some $\delta_0 > \frac{1}{2}$ the following asymptotical representation holds:*

$$\begin{aligned} L_k f(x) &= c_n \frac{e^{ik|x|} k^{\frac{n-3}{2}}}{|x|^{\frac{n-1}{2}}} \int_{R^n} e^{-ik(\theta', y)} \left(i\nabla(\vec{W}(y)f) + i\vec{W}(y)\nabla f - \tilde{q}(y)f \right) dy + \\ &+ o\left(\frac{1}{|x|^{\frac{n-1}{2}}}\right), \quad |x| \rightarrow \infty, \end{aligned} \quad (2.5)$$

where $\theta' = \frac{x}{|x|}$.

Proof. In view of (1.9) one must study the behavior for $|x| \rightarrow \infty$ of the function

$$G_k^+(|x - y|) = \frac{i}{4} \left(\frac{k}{2\pi|x - y|} \right)^{\frac{n-2}{2}} H_{\frac{n-2}{2}}^{(1)}(k|x - y|),$$

where $H_{\frac{n-2}{2}}^{(1)}$ denotes the Hankel function of first kind and of order $\frac{n-2}{2}$. In order to do that we take two cases, $k|x - y| > 1$ and $k|x - y| < 1$. For the first case we use the behavior of the Hankel function $H_{\frac{n-2}{2}}^{(1)}$ for large argument (see, for example, [5]), i.e.

$$H_{\frac{n-2}{2}}^{(1)}(z) = c_n \frac{e^{iz}}{\sqrt{z}} + O\left(\frac{1}{z^{\frac{3}{2}}}\right), \quad z \rightarrow +\infty.$$

So that ($k > 0$ is fixed) we have that,

$$G_k^+(k|x-y|) = c_n \frac{e^{ik|x-y|} k^{\frac{n-3}{2}}}{|x-y|^{\frac{n-1}{2}}} + O\left(\frac{1}{|x-y|^{\frac{n+1}{2}}}\right), \quad |x| \rightarrow +\infty.$$

Consider two subcases, $|y| \leq |x|^a$ and $|y| \geq |x|^a$, where $0 < a < \frac{1}{2}$ is a parameter. In the first case we have (since $a < \frac{1}{2}$)

$$|x-y|^{-\frac{n-1}{2}} = |x|^{-\frac{n-1}{2}} (1 + O(|x|^{a-1})), \quad |x| \rightarrow +\infty.$$

That is why we have for $|y| \leq |x|^a$ with $0 < a < \frac{1}{2}$ that

$$|x-y|^{-\frac{n-1}{2}} e^{ik|x-y|} = \frac{e^{ik|x|} e^{-ik(\theta', y)}}{|x|^{\frac{n-1}{2}}} + O(|x|^{\frac{n+1}{2}-2a}), \quad \theta' = \frac{x}{|x|},$$

as $|x| \rightarrow +\infty$. Substituting this asymptotic to the integral

$$S_1 := \int_{k|x-y|>1} G_k^+(|x-y|) \left(i\nabla(\vec{W}(y)f) + i\vec{W}(y)\nabla f - \tilde{q}(y)f \right) dy$$

gives that

$$\begin{aligned} S_1 &= \int_{k|x-y|>1, |y|\leq|x|^a} G_k^+(|x-y|) \left(i\nabla(\vec{W}(y)f) + i\vec{W}(y)\nabla f - \tilde{q}(y)f \right) dy + \\ &+ \int_{k|x-y|>1, |y|\geq|x|^a} G_k^+(|x-y|) \left(i\nabla(\vec{W}(y)f) + i\vec{W}(y)\nabla f - \tilde{q}(y)f \right) dy = \\ &= c_n \frac{e^{ik|x|} k^{\frac{n-3}{2}}}{|x|^{\frac{n-1}{2}}} \int_{k|x-y|>1, |y|\leq|x|^a} e^{-ik(\theta', y)} \left(i\nabla(\vec{W}(y)f) + i\vec{W}(y)\nabla f - \tilde{q}(y)f \right) dy + \\ &+ \int_{k|x-y|>1, |y|\geq|x|^a} G_k^+(|x-y|) \left(i\nabla(\vec{W}(y)f) + i\vec{W}(y)\nabla f - \tilde{q}(y)f \right) dy + \\ &+ \int_{k|x-y|>1, |y|\leq|x|^a} O(|x|^{\frac{n+1}{2}-2a}) \left(i\nabla(\vec{W}(y)f) + i\vec{W}(y)\nabla f - \tilde{q}(y)f \right) dy. \quad (2.6) \end{aligned}$$

The conditions (2.1)-(2.3) allow us easily conclude that for any $f \in H_{-\delta_0}^1(R^n)$ the integrand $i\nabla(\vec{W}(y)f) + i\vec{W}(y)\nabla f - \tilde{q}(y)f$ belongs to $L^1(R^n)$. Hence, the last term in the latter sum is $o(|x|^{-\frac{n-1}{2}})$ since $0 < a < \frac{1}{2}$. Denoting the second

term in the latter sum (2.6) by I we may estimate it (using conditions (2.3)) as follows:

$$\begin{aligned}
|I| &\leq C \int_{k|x-y|>1, |y|\geq|x|^a} \frac{|i\nabla(\vec{W}(y)f) + i\vec{W}(y)\nabla f - \tilde{q}(y)f|}{|x-y|^{\frac{n-1}{2}}} dy \leq \\
&\leq \frac{C}{|x|^{\frac{n-1}{2}}} \int_{|x|^a \leq |y| \leq \frac{|x|}{2}} |i\nabla(\vec{W}(y)f) + i\vec{W}(y)\nabla f - \tilde{q}(y)f| dy + \\
&\quad + C \int_{|y|\geq \frac{|x|}{2}} \frac{|\nabla\vec{W}(y)||f| + |\vec{W}(y)||\nabla f| + |\tilde{q}(y)||f|}{|x-y|^{\frac{n-1}{2}}} dy \leq \\
&\leq o\left(\frac{1}{|x|^{\frac{n-1}{2}}}\right) + C \int_{|y|\geq \frac{|x|}{2}} \frac{|f| + |\nabla f|}{|x-y|^{\frac{n-1}{2}}|y|^\mu} dy.
\end{aligned}$$

Since $f \in H_{-\delta_0}^1(R^n)$ the latter inequality implies

$$|I| \leq o\left(\frac{1}{|x|^{\frac{n-1}{2}}}\right) + \frac{C}{|x|^{\frac{n-1}{2}}} \left(\int_{|y|\geq \frac{|x|}{2}} \frac{1}{|x-y|^{n-1}|y|^{2\mu-(n-1)-2\delta_0}} dy \right)^{\frac{1}{2}} \|f\|_{H_{-\delta_0}^1(R^n)}. \quad (2.7)$$

Since $\mu > \frac{n+1}{2}$ then $\delta_0 > \frac{1}{2}$ can be chosen here such that the last integral in (2.7) might be considered as the convolution of "weak singularities" and therefore we have

$$I = o\left(\frac{1}{|x|^{\frac{n-1}{2}}}\right). \quad (2.8)$$

The first case $k|x-y| > 1$ is thus completely investigated.

In order to consider the second case $k|x-y| < 1$ we use the behavior of the Hankel function $H_{\frac{n-2}{2}}^{(1)}$ for small argument (see [5])

$$H_{\frac{n-2}{2}}^{(1)}(z) = \begin{cases} cz^{-\frac{n-2}{2}}, & n > 2, \\ c(1 + \log(z)), & n = 2, \end{cases} \quad z \rightarrow 0.$$

Let us consider first $n = 3$. Then using this asymptotic and taking into account that for fixed $k > 0$ and large $|x|$ it can be assumed that $|y| \geq \frac{|x|}{2}$, one can estimate the integral

$$S_2 := \int_{k|x-y|<1} G_k^+(|x-y|) \left(i\nabla(\vec{W}(y)f) + i\vec{W}(y)\nabla f - \tilde{q}(y)f \right) dy$$

as

$$\begin{aligned}
|S_2| &\leq C \left(\int_{k|x-y|<1} |G_k^+(k|x-y|)|^2 dy \right)^{\frac{1}{2}} \left(\int_{|y|\geq \frac{|x|}{2}} \frac{|\nabla f|^2 + |f|^2}{|y|^{2\mu}} dy \right)^{\frac{1}{2}} \\
&\leq \frac{C}{|x|^{\mu-\delta_0}} \left(\int_{k|x-y|<1} |x-y|^{-2} dy \right)^{\frac{1}{2}} \|f\|_{H_{-\delta_0}^1(R^n)} = o\left(\frac{1}{|x|}\right), \quad (2.9)
\end{aligned}$$

for $|x| \rightarrow +\infty$, if δ_0 is chosen such that $\frac{1}{2} < \delta_0 < 1$. In the same way we have that

$$S_2 = o\left(\frac{1}{|x|^{\frac{1}{2}}}\right), \quad |x| \rightarrow +\infty$$

in the two dimensional case due to the behavior of the Hankel function $H_0^{(1)}$ for small argument. Combining (2.6), (2.8) and (2.9) we get (2.5).

Remark 2.1. *The proof of the last lemma shows that the function $L_k f(x)$ is continuous for all x such that $|x| \geq R$, where R is large enough.*

These lemmas allow us to obtain the main result of this work.

Theorem 2.1. *Under the same assumptions as in Lemma 2.2 and for any $k \neq 0$ the integral equation (1.6) has a unique scattering solution from the space $H_{-\delta_0}^1(R^n)$ for some $\delta_0 > \frac{1}{2}$.*

Proof. Since the operator L_k is compact in the space $H_{-\delta_0}^1(R^n)$ we can apply the Riesz theory in this Hilbert space. Based on this methodology we will prove that the homogeneous equation $f - L_k f = 0$ has only the trivial solution in the space $H_{-\delta_0}^1(R^n)$ (i.e. the operator $I - L_k$ is injective). But this will imply that the operator is also surjective and the inverse $(I - L_k)^{-1}$ is bounded in $H_{-\delta_0}^1(R^n)$. This condition is equivalent to the claim that the equation (1.6) has a unique solution from the space $H_{-\delta_0}^1(R^n)$.

It is possible to check that any $f \in H_{-\delta_0}^1(R^n)$ which satisfies the homogeneous equation $f - L_k f = 0$ belongs to $H_{loc}^2(R^n)$ and satisfies also the equation

$$Hf = k^2 f$$

and Sommerfeld radiation condition (1.4). These facts imply that

$$0 = \int_{|x| \leq R} \nabla \left(\bar{f}(\nabla + i\vec{W})f - f(\nabla - i\vec{W})\bar{f} \right) dx.$$

The divergence theorem then gives that

$$0 = \int_{|x|=R} (\bar{f} \partial_\nu f - f \partial_\nu \bar{f}) d\sigma(x) + 2i \int_{|x|=R} |f|^2 \vec{W} \nu d\sigma(x),$$

where ν denotes the normal vector at the boundary of the ball B_R . The radiation conditions (1.4) for the functions f and \bar{f} allow us to conclude that

$$0 = 2i \int_{|x|=R} |f|^2 \vec{W} \nu d\sigma(x) + 2ik \int_{|x|=R} |f|^2 d\sigma(x) + o\left(\frac{1}{R^{\frac{n-1}{2}}}\right) \int_{|x|=R} f d\sigma(x).$$

Now the assumption that the function $f \in H_{-\delta_0}^1(R^n)$ satisfies the homogeneous equation $f - L_k f = 0$ implies by the equation (2.5) that

$$f = O\left(\frac{1}{|x|^{\frac{n-1}{2}}}\right), \quad |x| \rightarrow \infty.$$

This behavior and the condition (2.3) imply that

$$\int_{|x|=R} |f|^2 d\sigma(x) = o(1), \quad R \rightarrow \infty.$$

This fact and Lemma 2.3 (see (2.5)) allow us easily conclude that (it is enough to integrate in (2.5) with respect to x)

$$\int_{R^n} e^{-ik(\theta', y)} \left(i \nabla(\vec{W}(y) f(y)) + i \vec{W}(y) \nabla f(y) - \tilde{q}(y) f(y) \right) dy = 0.$$

Thus, we have actually (since the function f satisfies the homogeneous equation $f - L_k f = 0$) that

$$f = o\left(\frac{1}{|x|^{\frac{n-1}{2}}}\right), \quad |x| \rightarrow \infty. \quad (2.10)$$

The Sommerfeld radiation condition (1.4) implies also that

$$\nabla f = o\left(\frac{1}{|x|^{\frac{n-1}{2}}}\right), \quad |x| \rightarrow \infty. \quad (2.11)$$

Using these two facts we are going to prove that $f = 0$ a.e. In order to do so we first prove that $f(x) \equiv 0$ for $|x| \geq R_0$ with R_0 large enough. Indeed, it suffices to prove that, for $r \geq R_0$, the radial function

$$F(r) := \int_{S^{n-1}} f(r\theta)\phi(\theta) d\theta \quad (2.12)$$

is identically equal to zero for each eigenfunction ϕ of the Laplace operator Δ_S on the unit sphere S^{n-1} . The eigenfunctions satisfy the equations

$$(\Delta_S + \mu^2)\phi = 0, \quad \mu^2 = k(k + n - 2), \quad k = 0, 1, 2, \dots,$$

where μ^2 are the eigenvalues of the Laplace operator Δ_S (see, for example, [24]).

In view of the formula for the Laplacian Δ on R^n in polar coordinates

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_S,$$

it follows that $F(r)$ satisfies the ordinary differential equation

$$\begin{aligned} F''(r) + \frac{n-1}{r} F'(r) + (k^2 - \frac{\mu^2}{r^2}) F(r) = \\ = -2i \int_{S^{n-1}} \vec{W}(r\theta) \nabla f(r\theta) \phi(\theta) d\theta - \int_{S^{n-1}} \tilde{V}(r\theta) f(r\theta) \phi(\theta) d\theta, \end{aligned} \quad (2.13)$$

where $\tilde{V} = i\nabla \vec{W} + |\vec{W}|^2 + V$. Let us rewrite this linear equation in the form

$$F''(r) + \frac{n-1}{r} F'(r) + (k^2 - \frac{\mu^2}{r^2}) F(r) = \Phi(r, F).$$

Since V and \vec{W} satisfy (2.3) then it is not difficult to check that function $\Phi(r)$ from the right-hand side of equation (2.13) for $r \rightarrow \infty$ behaves as

$$\Phi(r, F) = O\left(\frac{1}{r^\alpha}\right), \quad \alpha > n. \quad (2.14)$$

It is well-known (see, for example, [5]) that the homogeneous equation corresponding to (2.13) has two linearly independent solutions $r^{-\frac{n-2}{2}} H_\nu^{(j)}(kr)$, $j = 1, 2$, where $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ are the Hankel functions of order ν , $\nu^2 = \mu^2 + (\frac{n-2}{2})^2$, and of the first and the second kind, respectively. In view of

the asymptotic behavior of the Hankel functions (see [5]) it follows that the behavior of these two solutions is of the form

$$r^{-\frac{n-2}{2}} H_\nu^{(j)}(kr) = \frac{C_j}{r^{\frac{n-1}{2}}} e^{\pm ikr} + o\left(\frac{1}{r^{\frac{n-1}{2}}}\right), \quad r \rightarrow \infty, \quad (2.15)$$

where $+$ corresponds to $j = 1$ and $-$ corresponds to $j = 2$. Next, we use Green's function (see, for example, [25]) for the Bessel equation with Dirichlet boundary conditions on the interval $[r_0, 2r_0]$, where r_0 is large enough,

$$g(r, \xi) = \frac{C}{\xi^{n-1}} \begin{cases} u_1(r)u_2(\xi), & r < \xi, \\ u_1(\xi)u_2(r), & r > \xi. \end{cases} \quad (2.16)$$

Here u_1 and u_2 are two linearly independent solutions of homogeneous equation (2.13) which satisfy homogeneous Dirichlet boundary conditions at r_0 and $2r_0$, respectively, that is,

$$u_1(r) = \left(\frac{1}{r_0 r}\right)^{\frac{n-2}{2}} (H_\nu^{(1)}(kr_0)H_\nu^{(2)}(kr) - H_\nu^{(1)}(kr)H_\nu^{(2)}(kr_0)),$$

$$u_2(r) = \left(\frac{1}{2r_0 r}\right)^{\frac{n-2}{2}} (H_\nu^{(1)}(2kr_0)H_\nu^{(2)}(kr) - H_\nu^{(1)}(kr)H_\nu^{(2)}(2kr_0)),$$

and $C = -r_0^{n-1}u_2(r_0)u_1'(r_0)$. Using this Green's function, equation (2.13) can be reduced to the following linear integral equation

$$F(r) = K_1 r^{-\frac{n-2}{2}} H_\nu^{(1)}(kr) + K_2 r^{-\frac{n-2}{2}} H_\nu^{(2)}(kr) + \int_{r_0}^{2r_0} g(r, \xi) \Phi(\xi, F) d\xi,$$

where K_1 and K_2 are constants. This integral equation can be uniquely solved by iterations. Since (2.14) and (2.15) hold then using representation (2.16) we obtain for the integral part of the latter integral equation the following estimate:

$$\int_{r_0}^{2r_0} g(r, \xi) \Phi(\xi, F) d\xi = O\left(\frac{1}{r^{\alpha-1}} + \frac{1}{r^{n-1}r_0^{\alpha-n}}\right), \quad \alpha > n, \quad (2.17)$$

as $r_0 \rightarrow +\infty$ and $r_0 \leq r \leq 2r_0$. Thus, the estimates (2.17) allow us to conclude that any solution $F(r)$ of the non-homogeneous equation (2.13) has asymptotic (when $r \rightarrow \infty$) that is the linear combination of two asymptotic (2.15). Since the hypothesis implies that $F(r) = o\left(\frac{1}{r^{\frac{n-1}{2}}}\right)$, we deduce that

$V(r) \equiv 0$ for all r large enough. Thus, the same is true for $f(x)$ (see definition (2.12)) for all $|x| \geq R_0$ with R_0 large enough.

We are in the position now to apply the unique continuation principle (UCP) in R^n . Due to UCP for a second order elliptic differential operators with real coefficients (see Theorem 17.2.8 in [3]) we may immediately conclude that $f \equiv 0$ in R^n . Thus, $I - L_k$ is injective and therefore, the integral equation (1.6) has a unique solution from the space $H_{-\delta_0}^1(R^n)$ which is given by the formula

$$u_{sc} = (I - L_k)^{-1} \tilde{u}_0 \iff u = u_0 + (I - L_k)^{-1} L_k u_0, \quad (2.18)$$

where u is as in (1.5). Theorem 2.1 is completely proved.

Corollary 2.1. *If the conditions (2.1)-(2.3) are satisfied then the magnetic Schrödinger operator H has no positive eigenvalues.*

Proof. If λ is a positive eigenvalue of H then

$$Hu = \lambda u, \quad u \in L^2(R^n), \quad Hu \in L^2(R^n).$$

It means that this u belongs to the domain of the Friedrichs self-adjoint extension of H and therefore $u \in W_2^1(R^n)$. This fact allows us to conclude that this u satisfies the homogeneous equation

$$u = L_{\sqrt{\lambda}} u$$

with an integral operator from (1.9). Thus, since $W_2^1(R^n) \subset H_{-\delta}^1(R^n)$ for $\delta > \frac{1}{2}$ (actually this imbedding holds for any $\delta > 0$) we may apply to this u the same proof as in Theorem 2.1 and conclude that actually $u \equiv 0$.

Using Agmon's results (1.8) the operator L_k can be extended to the space $L_{-\delta}^2(R^n)$ as a uniformly bounded operator with respect to $k \geq 1$ such that

$$\|u_{sc}\|_{L_{-\delta}^2(R^n)} \leq C, \quad k \geq 1, \quad (2.19)$$

where constant C depends only on the corresponding norms of V , \vec{W} and $\nabla \vec{W}$. Based on this fact one can show that if the corresponding norms of V , \vec{W} and $\nabla \vec{W}$ are small enough then the operator norm of L_k as an operator from $L_{-\delta}^2(R^n)$ to itself is strictly less than 1. In that case the formula (2.18) can be rewritten as

$$u = u_0 + \sum_{j=1}^{\infty} L_k^j(u_0). \quad (2.20)$$

Thus, the scattering solution u can be obtained as the series of iterations of u_0 in the equation (1.5).

3 Scattering amplitude and direct backscattering Born approximation

In this section we will consider the direct backscattering Born approximation for the magnetic Schrödinger operator H with conditions (2.1)-(2.3). The motivation to this problem is connected to the fact that the knowledge of the scattering amplitude with the backscattering data gives essential information about the unknown function V and \vec{W} .

Theorem 2.1 and Lemma 2.3 (see (2.5)) yield the following asymptotical representation for the scattering solutions $u(x, k, \theta)$ with fixed $k > 0$ as $|x| \rightarrow +\infty$:

$$u(x, k, \theta) = e^{ik(x, \theta)} + c_n \frac{e^{ik|x|} k^{\frac{n-3}{2}}}{|x|^{\frac{n-1}{2}}} A(k, \theta', \theta) + o\left(\frac{1}{|x|^{\frac{n-1}{2}}}\right),$$

where function A is called the scattering amplitude and defined by

$$A(k, \theta', \theta) = \int_{R^n} e^{-ik(\theta', y)} \left(i \nabla(\vec{W}(y)u) + i \vec{W}(y) \nabla u - \tilde{q}(y)u \right) dy. \quad (3.1)$$

Substituting $u = u_0 + u_{sc}$ into the equation (3.1) gives that

$$\begin{aligned} A(k, \theta', \theta) &= \int_{R^n} e^{-ik(\theta', y)} \left(i \nabla(\vec{W}(y)u_0) + i \vec{W}(y) \nabla u_0 - \tilde{q}(y)u_0 \right) dy + \\ &+ \int_{R^n} e^{-ik(\theta', y)} \left(i \nabla(\vec{W}(y)u_{sc}) + i \vec{W}(y) \nabla u_{sc} - \tilde{q}(y)u_{sc} \right) dy := \\ &:= A_B(k, \theta', \theta) + R(k, \theta', \theta). \end{aligned} \quad (3.2)$$

The function A_B is called the direct Born approximation. It can be checked (using integration by parts) that A_B is actually equal to

$$A_B(k, \theta', \theta) = -k(\theta + \theta') F(\vec{W})(k(\theta - \theta')) - F(\tilde{q})(k(\theta - \theta')),$$

where F denotes usual n -dimensional Fourier transform as

$$F(f)(\xi) = \int_{R^n} f(x) e^{i(x, \xi)} dx.$$

The particular case $\theta' = -\theta$ yields the direct backscattering Born approximation

$$A_B^b(k, -\theta, \theta) = -F(\tilde{q})(2k\theta). \quad (3.3)$$

Formulae (3.2) and (3.3) show that in the frame of the Born approximation

$$A(k, -\theta, \theta) \approx -F(|\vec{W}|^2 + V)(2k\theta).$$

But we want to write more terms in the Born series. For this purpose we calculate term R in the scattering amplitude. Using the series (2.20) and integration by parts we have that

$$\begin{aligned} R(k, -\theta, \theta) = & -i \int_{R^n} e^{ik(\theta, y)} \nabla \vec{W}(y) L_k u_0(y) dy + 2k\theta \int_{R^n} e^{ik(\theta, y)} \vec{W}(y) L_k u_0(y) dy - \\ & - \int_{R^n} e^{ik(\theta, y)} \tilde{q}(y) L_k u_0(y) dy + R_2(k, -\theta, \theta), \end{aligned}$$

where the term R_2 corresponds to the series $\sum_{j=2}^{\infty} L_k^j(u_0)$ and equals to

$$\begin{aligned} R_2(k, -\theta, \theta) = & i \int_{R^n} e^{ik(\theta, y)} \nabla \vec{W}(y) \sum_{j=2}^{\infty} L_k^j u_0(y) dy + \\ & + 2i \int_{R^n} e^{ik(\theta, y)} \vec{W}(y) \nabla \left(\sum_{j=2}^{\infty} L_k^j u_0(y) \right) dy - \int_{R^n} e^{ik(\theta, y)} \tilde{q}(y) \sum_{j=2}^{\infty} L_k^j u_0(y) dy. \end{aligned} \tag{3.4}$$

It will be shown that the term which correspond to R_2 in the definition (3.1) might be neglected because of the smallness of the operator norm L_k in the space $L^2_{-\delta}(R^n)$.

Since $L_k u_0(y)$ is equal to

$$\int_{R^n} e^{ik(\theta, z)} G_k^+(|y - z|) \left(i \nabla \vec{W}(z) - 2k\theta \vec{W}(z) - \tilde{q}(z) \right) dz,$$

then we obtain (after some simple calculations) the following representation:

$$\begin{aligned} R(k, -\theta, \theta) := & R_1(k, -\theta, \theta) + R_2(k, -\theta, \theta) = \\ = & \int_{R^n} \int_{R^n} e^{ik(\theta, y+z)} G_k^+(|y - z|) \nabla \vec{W}(y) \nabla \vec{W}(z) dy dz + \\ & + 4ik \int_{R^n} \int_{R^n} e^{ik(\theta, y+z)} G_k^+(|y - z|) \nabla \vec{W}(y) \theta \vec{W}(z) dy dz - \end{aligned}$$

$$\begin{aligned}
& -4k^2 \int_{R^n} \int_{R^n} e^{ik(\theta, y+z)} G_k^+(|y-z|) \theta \vec{W}(y) \theta \vec{W}(z) dy dz + \\
& + \int_{R^n} \int_{R^n} e^{ik(\theta, y+z)} G_k^+(|y-z|) \tilde{q}(y) \tilde{q}(z) dy dz + R_2 := \\
& := I_1 + I_2 + I_3 + I_4 + R_2.
\end{aligned} \tag{3.5}$$

It can be mentioned here that this equality must be understood in the sense of tempered distributions.

Using the facts $F(G_k^+)(\eta) = \frac{1}{\eta^2 - k^2 - i0}$ and $F(\phi \cdot \psi) = (2\pi)^{-n} F(\phi) * F(\psi)$ we can calculate the terms $I_j, j = 1, 2, 3, 4$, more precisely as

$$\begin{aligned}
I_1 &= (2\pi)^{-n} \int_{R^n} \frac{F(\nabla \vec{W})(k\theta + \eta) F(\nabla \vec{W})(k\theta - \eta)}{\eta^2 - k^2 - i0} d\eta, \\
I_2 &= 4ik(2\pi)^{-n} \int_{R^n} \frac{F(\nabla \vec{W})(k\theta + \eta) \theta F(\vec{W})(k\theta - \eta)}{\eta^2 - k^2 - i0} d\eta, \\
I_3 &= -4k^2(2\pi)^{-n} \int_{R^n} \frac{\theta F(\vec{W})(k\theta + \eta) \theta F(\vec{W})(k\theta - \eta)}{\eta^2 - k^2 - i0} d\eta, \\
I_4 &= (2\pi)^{-n} \int_{R^n} \frac{F(\tilde{q})(k\theta + \eta) F(\tilde{q})(k\theta - \eta)}{\eta^2 - k^2 - i0} d\eta.
\end{aligned} \tag{3.6}$$

Our next step is to neglect the term R_2 in (3.5) and justify this neglect. Indeed, using (1.7) and Agmon's results (1.8) the $L^2_{-\delta}$ -norm of $L_k u_0$ can be estimated as (uniformly in $|k| \geq 1$)

$$\|L_k u_0\|_{L^2_{-\delta}(R^n)} \leq c \left(\|\nabla \vec{W}\|_{L^2_{\delta}(R^n)} + \|\vec{W}\|_{L^2_{\delta}(R^n)} + \|\tilde{q}\|_{L^2_{\delta}(R^n)} \right).$$

The conditions (2.1)-(2.3) show that the right hand-side of the latter inequality is finite. Thus, there is a constant c_0 depending only on the L^2_{δ} -norms of functions \vec{W} , $\nabla \vec{W}$ and \tilde{q} such that uniformly in $|k| \geq 1$

$$\|L_k u_0\|_{L^2_{-\delta}(R^n)} \leq c_0. \tag{3.7}$$

At the same time for any function $f \in H^1_{-\delta}(R^n)$ with some $\delta > \frac{1}{2}$ we can easily obtain

$$\|L_k f\|_{L^2_{-\delta}(R^n)} \leq c \left(\|\nabla \vec{W}\|_{L^p_{2\delta}(R^n)} + \|\vec{W}\|_{L^\infty_{2\delta}(R^n)} + \|\tilde{q}\|_{L^p_{2\delta}(R^n)} \right) \|f\|_{H^1_{-\delta}(R^n)}, \tag{3.8}$$

where p is the same as in the conditions (2.1)-(2.2). We can rewrite (3.8) in the form of operator norm

$$\|L_k\|_{H_{-\delta}^1(R^n) \rightarrow L_{-\delta}^2(R^n)} \leq c_1, \quad (3.9)$$

where constant c_1 depends only on the norms of functions \vec{W} , $\nabla \vec{W}$ and \tilde{q} from (3.8). Hence, we may assume that these norms are chosen so small that $c_1 < 1$. Now we extend the operator L_k as an operator from $L_{-\delta}^2(R^n)$ to $L_{-\delta}^2(R^n)$ with the same norm estimate as in (3.9). This fact together with estimate (3.7) imply that

$$\left\| \sum_{j=2}^{\infty} L_k^j u_0 \right\|_{L_{-\delta}^2(R^n)} \leq \frac{c_0 c_1}{1 - c_1}. \quad (3.10)$$

Hence, the left hand-side of (3.10) can be made as small as we want if c_1 (and, in addition, c_0) are chosen small enough. This fact and duality arguments show that

$$\left\| \nabla \left(\sum_{j=2}^{\infty} L_k^j u_0 \right) \right\|_{H_{-\delta}^{-1}(R^n)} \leq \frac{c_0 c_1}{1 - c_1}, \quad (3.11)$$

The estimates (3.10) and (3.11) imply that one can have the term $R_2(k, -\theta, \theta)$ as small as desired uniformly in $|k| \geq 1$ and $\theta \in S^{n-1}$ if the corresponding norms of functions \vec{W} , $\nabla \vec{W}$ and \tilde{q} (or the constants c_0 and c_1) are chosen small enough. Thus, the term $R_2(k, -\theta, \theta)$ can be neglected in the Born approximation.

Summarizing our considerations (see (3.5)-(3.6) and (3.10)-(3.11)) we may now obtain the following direct backscattering Born approximation (more precise than (3.3)) for the magnetic Schrödinger operator

$$A(k, -\theta, \theta) \approx -F(|\vec{W}|^2 + V)(2k\theta) + I_1 + I_2 + I_3 + I_4. \quad (3.12)$$

This formula gives us very good approximation for the backscattering amplitude A . It is very important that for this approximation we need to have only the magnetic potential \vec{W} and electric potential V , but we do not need (as we can see the formula (3.1)) to have the scattering solutions $u(x, k, \theta)$ of the equation

$$Hu(x) = k^2 u(x).$$

This direct approximation (3.12) will be effectively used for the inverse backscattering Born approximation. Namely, due to formulas (3.6) we will be able to calculate precisely the quadratic term in the Born series that corresponds to the inverse backscattering approximation and to estimate its

smoothness. This smoothness result together with (3.12) will give us the solution of the inverse backscattering problem with respect to the reconstruction of the singularities and jumps of the unknowns. These problems will be investigated carefully in the futures publications.

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